

THE FILTER DICHOTOMY PRINCIPLE DOES NOT IMPLY THE SEMIFILTER TRICHOTOMY PRINCIPLE

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ABSTRACT. We answer Blass' question from 1989 of whether the inequality $\mathfrak{u} < \mathfrak{g}$ is strictly stronger than the filter dichotomy principle [4, page 36] affirmatively. We show that there is a forcing extension in which every non-meagre filter on ω is ultra by finite-to-one and the semifilter trichotomy does not hold. This trichotomy says: every semifilter is either meagre or comeagre or ultra by finite-to-one. The trichotomy is equivalent to the inequality $\mathfrak{u} < \mathfrak{g}$ by work of Blass and Laflamme. Combinatorics of block sequences is used to establish forcing notions that preserve suitable properties of block sequences.

1. INTRODUCTION

We separate two useful combinatorial principles: We show the filter dichotomy principle is strictly weaker than the semifilter trichotomy principle. Consequences of the latter and equivalent statements to the latter in the realm of measure, category, rarefaction orders are investigated in [21, 17, 5]. Paul Larson proves in [22] a long-standing question about medial limits: The filter dichotomy implies that there are none. Our result on the combinatorial side thus separates some very powerful principles in analysis.

We first recall the definitions: For $B \subseteq \omega$ and $f: \omega \rightarrow \omega$, we let $f''B = \{f(b) : b \in B\}$ and $f^{-1}B = \{n : f(n) \in B\}$. By a *filter* we mean a proper filter on ω . We call a filter *non-principal* if it contains all cofinite sets. Let \mathcal{F} be a non-principal filter on ω and let $f: \omega \rightarrow \omega$ be finite-to-one (that means that the preimage of each natural number is finite). Then also $f(\mathcal{F}) = \{X : f^{-1}X \in \mathcal{F}\}$ is a non-principal filter. It is the filter generated by $\{f''X : X \in \mathcal{F}\}$. From now on we consider only non-principal filters. Two filters \mathcal{F} and \mathcal{G} are *nearly coherent*, if there is some finite-to-one $f: \omega \rightarrow \omega$ such that $f(\mathcal{F}) \cup f(\mathcal{G})$ generates a filter. We also say to this situation that $f(\mathcal{F})$ and $f(\mathcal{G})$ are *coherent*. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. A *semifilter* \mathcal{S} is a subset of $[\omega]^\omega$ that contains ω as an element and that is closed under almost supersets, i.e., $(\forall X \in \mathcal{S})(\forall Y \in [\omega]^\omega)(X \setminus Y \text{ finite} \rightarrow Y \in \mathcal{S})$. In particular, $[\omega]^\omega$ is a semifilter.

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The *filter dichotomy principle*, abbreviated FD, says that for every filter there is a finite-to-one function f such that $f(\mathcal{F})$ is either the filter of cofinite sets (also called the *Fréchet filter*) or an ultrafilter. In the latter case we call \mathcal{F} *ultra by finite-to-one* or *nearly ultra*. A semifilter \mathcal{S} is called *meagre/comeagre* if the set of the characteristic functions of the members of \mathcal{S} is a meagre/comeagre subset of the space 2^ω .

We recall a connection between meagreness and unboundedness in the eventual domination order. Let ${}^\omega\omega$ denote the set of functions from ω to ω . For $f, g \in {}^\omega\omega$ we say g eventually dominates f and write $f \leq^* g$ iff $(\exists n_0 \in \omega)(\forall n \geq n_0)(f(n) \leq g(n))$. Talagrand [34] showed

Lemma 1.1. *For every semifilter \mathcal{S} the following are equivalent*

- (1) *There is a finite-to-one function such that $\{X : (\exists S \in \mathcal{S})(f''S \subseteq X)\}$ is the Fréchet filter.*
- (2) *\mathcal{S} is meagre.*
- (3) *The set of enumerating functions of members of \mathcal{S} is \leq^* -bounded.*

The *semifilter trichotomy principle*, abbreviated SFT, says that for every semifilter \mathcal{S} either \mathcal{S} is meagre or $f(\mathcal{S})$ is ultra or $f(\mathcal{S}) = [\omega]^\omega$ for some finite-to-one f . The latter is equivalent to \mathcal{S} being comeagre, for an explicit proof see [22, Th. 4.1].

The semifilter trichotomy can also be formulated in terms of two cardinal characteristics: Let \mathcal{F} be a filter on ω . $\mathcal{B} \subseteq \mathcal{F}$ is a *base* for \mathcal{F} if for every $X \in \mathcal{F}$ there is some $Y \in \mathcal{B}$ such that $Y \subseteq X$. The character of \mathcal{F} , $\chi(\mathcal{F})$, is the smallest cardinality of a base of \mathcal{F} . The cardinal \mathfrak{u} is the smallest character of a non-principal ultrafilter. We denote by \mathfrak{g} be the *groupwise density number*, that is the smallest number of groupwise dense sets whose intersection is empty. A set $\mathcal{G} \subseteq [\omega]^\omega$ is called *groupwise dense* if it is closed under almost subsets and for every strictly increasing function $f: \omega \rightarrow \omega$ there is an infinite A such that $\bigcup_{n \in A} [f(n), f(n+1)) \in \mathcal{G}$. Laflamme [21, Theorem 8] showed that $\mathfrak{u} < \mathfrak{g}$ implies SFT, and Blass [5] showed that SFT implies $\mathfrak{u} < \mathfrak{g}$. The purpose of this paper is to show the following:

Main Theorem. “FD and the negation of SFT” is consistent relative to ZFC.

A groupwise dense family that is closed under finite unions is called a groupwise dense ideal. The *groupwise density number for filters*, \mathfrak{g}_f , is the smallest number of groupwise dense ideals with empty intersection. From [8] and Blass [5], just read for groupwise dense ideals, it follows that $\mathfrak{u} < \mathfrak{g}_f$ is equivalent to FD. Moreover, FD implies $\mathfrak{b} = \mathfrak{u} < \mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$ [5]. Hence FD and not SFT is equivalent to $\mathfrak{g} \leq \mathfrak{u} < \mathfrak{g}_f$. Brendle [12] constructed a c.c.c. extension with $\kappa = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{b} = \kappa^+$, and asked whether $\mathfrak{b} = \mathfrak{g} < \mathfrak{g}_f$ is consistent. By Shelah’s $\mathfrak{g}_f \leq \mathfrak{b}^+$ in ZFC [33], the only constellation for $\mathfrak{b} \leq \mathfrak{g} < \mathfrak{g}_f$ is $\mathfrak{b} = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{b}^+$. Since in any model of the dichotomy and $\mathfrak{u} \geq \mathfrak{g}$ we have the cardinal constellation $\mathfrak{b} = \mathfrak{u} = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{c}$, the

main theorem also answers a question by Brendle [12, Question 10] about separating \mathfrak{g} and \mathfrak{g}_f above \mathfrak{b} in the \aleph_1 - \aleph_2 -scenario.

For $S, X \in [\omega]^\omega$ we say S splits X iff $X \cap S$ and $X \setminus S$ are both infinite. A set $SP \subseteq [\omega]^\omega$ is called *splitting* or a splitting family iff for every $X \in [\omega]^\omega$ there is some $S \in SP$ splitting X . The smallest cardinal of a splitting family is called the *splitting number* and denoted by \mathfrak{s} . Necessarily the splitting number \mathfrak{s} must be bounded by \mathfrak{u} for FD and $\mathfrak{u} \geq \mathfrak{g}$, because by [24, Cor. 4.4.], FD together with $\mathfrak{s} > \mathfrak{u}$ implies $\mathfrak{u} < \mathfrak{g}$.

The same argument shows:

Proposition 1.2. $\mathfrak{g}_f \leq \mathfrak{s}$ implies $\mathfrak{g} = \mathfrak{g}_f$.

Proof. Assume that we have groupwise dense families \mathcal{G}_α , $\alpha < \kappa$ for some $\kappa < \mathfrak{g}_f$. Then there is a diagonalisation D of the generated ideals, that is for every $\alpha < \kappa$ there are $A_{\alpha,i} \in \mathcal{G}_\alpha$, $i \leq n_\alpha$ such that $D \subseteq A_{\alpha,0} \cup \dots \cup A_{\alpha,n_\alpha}$. Then if $\kappa < \mathfrak{s}$ these $A_{\alpha,i} \cap D$ are not a splitting family on $[D]^\omega$ and hence there is some infinite $D' \subseteq D$ and there are i_α , $\alpha < \kappa$, such that $(\forall \alpha < \kappa)(D' \subseteq A_{\alpha,i_\alpha})$. So $D' \in \bigcap_{\alpha < \kappa} \mathcal{G}_\alpha$ and $\mathfrak{g} > \kappa$. \square

P -points and some cardinal characteristics are involved in our forcing construction. We recall some definitions: We say “ A is almost a subset of B ” and write $A \subseteq^* B$ iff $A \setminus B$ is finite. Similarly, the symbol $=^*$ denotes equality up to finitely many exceptions in $[\omega]^\omega$ or in ${}^\omega\omega$, the set of functions from ω to ω .

An ultrafilter \mathcal{U} is called a P -point if for every $\gamma < \omega_1$, for every $A_i \in \mathcal{U}$, $i < \gamma$, there is some $A \in \mathcal{U}$ such that for all $i < \gamma$, $A \subseteq^* A_i$; such an A is called a *pseudo-intersection* or a *diagonalisation* of the A_i , $i < \gamma$. Let \mathbb{P} be a notion of forcing. We say that \mathbb{P} preserves an ultrafilter \mathcal{U} if $\Vdash_{\mathbb{P}} “(\forall X \in [\omega]^\omega)(\exists Y \in \mathcal{U})(Y \subseteq X \vee Y \subseteq \omega \setminus X)”$ and in the contrary case we say “ \mathbb{P} destroys \mathcal{U} ”. If \mathbb{P} is proper and preserves \mathcal{U} and \mathcal{U} is a P -point, then \mathcal{U} stays a P -point [9, Lemma 3.2].

The only models of FD that have been known so far are also models of $\mathfrak{u} < \mathfrak{g}$ (and hence SFT). A ground model with CH is extended by an iterated forcing $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \omega_2, \alpha < \omega_2 \rangle$ that is built in the usual way: The iterand \mathbb{Q}_α is a \mathbb{P}_α -name and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$, and at limits we build \mathbb{P}_α with countable supports. The iterands are proper forcings that preserve at least one, indeed any P -point, and thus keep \mathfrak{u} small. The principle of near coherence of filters says that for any two filters (recall: they contain the Fréchet filter) are nearly coherent. An equivalent formulation is that any two non-principal ultrafilters are nearly coherent. The filter dichotomy implies NCF by [8]. If $\mathfrak{u} < \mathfrak{d}$, which follows from NCF, then by Ketonen [19] every filter witnessing \mathfrak{u} is a P -point, so we do not have to worry whether there is some non- P ultrafilter with a base of a smaller size. Let us write \mathbf{V}_α for $\mathbf{V}^{\mathbb{P}_\alpha}$, an arbitrary extension by a \mathbb{P}_α -generic filter G_α . Although \mathfrak{u} is kept small, at least at stationarily many limit steps $\alpha < \omega_2$ of cofinality ω_1

the next iterand adds a real that has supersets in all groupwise dense sets in $\mathbf{V}^{\mathbb{P}_\alpha} = \mathbf{V}_\alpha$ and thus $\mathfrak{g} = \aleph_2$ [8].

Some types of such models of $\mathfrak{u} < \mathfrak{g}$ are known: an iteration of length ω_2 with countable support of Blass–Shelah forcing over a ground model of CH [9] gives $\aleph_1 = \mathfrak{u} < \mathfrak{s} = \mathfrak{g} = 2^{\aleph_0} = \aleph_2$ and an iteration of length ω_2 with countable support of Miller forcing over a ground model of CH [10] gives $\aleph_1 = \mathfrak{u} = \mathfrak{s} < \mathfrak{g} = 2^{\aleph_0} = \aleph_2$. A third type of model of $\mathfrak{u} < \mathfrak{g}$ is given by a countable support iteration of Matet forcing [4]. Other proper tree forcings that preserve P -points can be interwoven into the iteration and, as long as at stationarily many steps of cofinality ω_1 a real is added that has a superset in each groupwise dense family in the intermediate model, the outcome is $\mathfrak{u} < \mathfrak{g}$.

The proof of our main theorem works with a construction that uses some techniques developed in Mildenerger and Shelah’s construction of a model of NCF and not FD in [26]. As there, we use Hindman’s theorem and Eisworth’s theorem on Matet forcing with stable ordered-union ultrafilters. We combine this with new work on Matet forcing with stable ordered union ultrafilters (also known as Milliken–Taylor ultrafilters). The main difficulty was to find the properties (I4) and (I5) of the iteration.

We give an overview over the construction: We let $S_i^2 = \{\alpha \in \omega_2 : \text{cf}(\alpha) = \omega_i\}$, $i = 0, 1$. A ground model with CH and $\diamond(S_1^2)$ is extended by a countable support iteration $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \omega_2, \alpha < \omega_2 \rangle$, with iterands of the form $\mathbb{Q}_\alpha = \mathbb{M}_{\mathcal{U}_\alpha}$. Alternatively, instead of assuming a diamond in the ground model we can force in a first forcing step with approximations $\langle \mathbb{P}_\beta, \mathbb{Q}_\alpha : \beta \leq \gamma, \alpha < \gamma \rangle$, $\gamma < \omega_2$. For such a representation of a forcing see [27, Section 2]. The tricky part of our proof is to find suitable stable ordered-union ultrafilters \mathcal{U}_α . In contrast to in [26] this time also the steps $\alpha \notin S_1^2$ have a more subtle influence and they must be gentler than the Blass–Shelah forcing, that increases the splitting number. The iterands are proper, indeed σ -centred iterands can be chosen. Their definition is based on ultrafilters and other names that are defined by induction. The essential properties of the iteration are:

First: As in [26], we preserve only one arbitrary P -point $\mathcal{E} \in \mathbf{V}_0$ that will be fixed forever, and we destroy many others. Eisworth [14] is the tool for preserving \mathcal{E} .

Second: We get FD with the aid of a diamond. A diamond sequence on S_1^2 is a sequence $\langle \mathcal{D}_\alpha : \alpha \in S_1^2 \rangle$ such that for any $X \subseteq \omega_2$ the set $\{\alpha \in S_1^2 : X \cap \alpha = \mathcal{D}_\alpha\}$ is stationary. If the diamond \mathcal{D}_α guesses a non-meagre filter $\mathcal{D}_\alpha[G] = \mathcal{F}$, we add with \mathbb{Q}_α a (generically flat) finite-to-one function f_α , such that $f_\alpha(\mathcal{F}) = f_\alpha(\mathcal{E})$ and \mathcal{E} stays an ultrafilter. Thus, by a reflection argument for countable support iterations of proper iterands of size $\leq \aleph_1$, FD holds in the final model.

Third: During our construction, a semifilter \mathcal{S} witnessing the failure of the semifilter trichotomy will be constructed as

$$(1.1) \quad \mathcal{S} = \{x \in [\omega]^\omega : (\exists \alpha \in \omega_2 \setminus S_1^2)(x \supseteq^* s_\alpha)\}$$

where the s_α is simply the \mathbb{Q}_α -generic real over $\mathbf{V}^{\mathbb{P}_\alpha} = \mathbf{V}_\alpha$ for $\alpha \in \omega_2$. The $\alpha \in \omega_2 \setminus S_1^2$ contribute to the name of \mathcal{S} , and for the $\alpha \in S_1^2$, s_α has an even more important role: To ensure that the iteration can be continued from \mathbb{P}_α to $\mathbb{P}_{\alpha+1}$. For this construction we use for $\alpha \in S_1^2$ Matet forcing \mathbb{M} thinned out with a Milliken–Taylor ultrafilter $\mathcal{U}_\alpha \in \mathbf{V}^{\mathbb{P}_\alpha}$, called $\mathbb{M}_{\mathcal{U}_\alpha}$, such that \mathcal{U}_α guarantees that our inductive construction can be carried on. Let \mathcal{R}^* stand for the set of finite-to-finite relations (explained in Section 4).

The aim is to build an iteration of length ω_2 with the following property:

- (I5) For all $\gamma \in \omega_2 \setminus S_1^2$:
 s_γ is \leq^* -unbounded over \mathbf{V}_γ and $(\forall \beta \in \gamma \setminus S_1^2)(\forall R \in \mathcal{R}^* \cap \mathbf{V}_\beta)(\mathbb{P}_{\gamma+1} \Vdash (s_\gamma \not\subseteq^* R s_\beta))$.

For this we will have to work quite a bit. An impatient knowledgeable reader might first want to read Lemma 5.1 in which we prove why (I5) together with other well-known properties of Matet forcing and of Matet forcing with suitable Milliken–Taylor ultrafilters ensures that $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ is a model of FD and not SFT.

A side remark: (I5) means that the counterexample \mathcal{S} is a semifilter consisting of somewhat solid sets s_α , $\alpha \in \omega_2 \setminus S_1^2$. Matet reals are more “solid” than Blass–Shelah reals: Matet blocks must stay or are dropped as a whole, whereas in Blass–Shelah forcing we can thin out each block according to a logarithmic measure. Both have very large complements, which is the reason that both diagonalise many families of groupwise dense sets. Matet forcing will be defined and explained in Section 2. Blass–Shelah forcing will not appear any more in this work. By Proposition 1.2 it is ruled out as a candidate for iterands.

Observation 1.3. *From the short proof of Laflamme’s [21] theorem $\mathfrak{u} < \mathfrak{g} \rightarrow \text{SFT}$ in [7, Lemma 9.15, Theorem 9.22] we read off the groupwise dense families that witness $\mathfrak{g} \leq \mathfrak{u}$ in our forcing extensions. In the ground model, fix a basis $\{E_\varepsilon : \varepsilon < \aleph_1\}$ for the P -point \mathcal{E} . Then we let*

$$(1.2) \quad \mathcal{G}_\varepsilon = \{Z \in [\omega]^\omega : (\exists S \in \mathcal{S})(\forall m, n \in Z) \\ ([m, n] \cap S \neq \emptyset \rightarrow [m, n] \cap E_\varepsilon \neq \emptyset) \wedge \\ (\exists T)(\omega \setminus T \in [\omega]^\omega \setminus \mathcal{S})(\forall m, n \in Z) \\ ([m, n] \cap T \neq \emptyset \rightarrow [m, n] \cap E_\varepsilon \neq \emptyset))\}.$$

Since \mathcal{S} is not meagre and not comeagre, the sets \mathcal{G}_ε are groupwise dense, and $\bigcap_{\varepsilon < \omega_1} \mathcal{G}_\varepsilon = \emptyset$ since \mathcal{S} is not equal to \mathcal{E} by finite-to-one.

The paper is organised as follows: In Section 2 we explain Matet forcing with centred systems and show how to preserve the non-meagreness and the density of a given set. In Section 3 we recall Matet forcing with stable

ordered-union ultrafilters and Eisworth's work. Section 4 introduces some notions for handling block sequences and finite-to-one functions. In Section 5 we define the iterated forcing orders that establish our consistency results.

Undefined notation on cardinal characteristics can be found in [2, 7]. Undefined notation about forcing can be found in [20, 32]. In the forcing, we follow the Israeli style that the stronger condition is the *larger* one. A very good background in proper forcing is assumed.

2. PRESERVING A NON-MEAGRE SET AND A DENSE SET

We define a variant of Matet forcing. For this purpose, we first introduce some notation about block-sequences. Our nomenclature follows Blass [3] and Eisworth [14].

We let \mathbb{F} be the collection of all finite non-empty subsets of ω . For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. A filter on \mathbb{F} is a subset of $\mathcal{P}(\mathbb{F})$ that is closed under intersections and supersets. A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of \mathbb{F} is called unmeshed if for all n , $a_n < a_{n+1}$. The set $(\mathbb{F})^\omega$ denotes the collection of all infinite unmeshed sequences in \mathbb{F} . If X is a subset of \mathbb{F} , we write $\text{FU}(X)$ for the set of all finite unions of members of X . We write $\text{FU}(\bar{a})$ instead of $\text{FU}(\{a_n : n \in \omega\})$.

Definition 2.1. *Given \bar{a} and \bar{b} in $(\mathbb{F})^\omega$, we say that \bar{b} is a condensation of \bar{a} and we write $\bar{b} \sqsubseteq \bar{a}$ if $\bar{b} \subseteq \text{FU}(\bar{a})$. We say \bar{b} is almost a condensation of \bar{a} and we write $\bar{b} \sqsubseteq^* \bar{a}$ iff there is an n such that $\langle b_t : t \geq n \rangle$ is a condensation of \bar{a} .*

We also call $\bar{b} \sqsubseteq^* \bar{a}$ a strengthening \bar{a} . We use the verb “to strengthen \bar{a} ” as an abbreviation for “to replace \bar{a} by an appropriate strengthening and call that strengthening again \bar{a} ”.

Definition 2.2. *A set $\mathcal{C} \subseteq (\mathbb{F})^\omega$ is called centred, if for any finite $C \subseteq \mathcal{C}$ there is $\bar{a} \in \mathcal{C}$ that is almost a condensation of any $\bar{c} \in C$.*

Definition 2.3. *In the Matet forcing, \mathbb{M} , the conditions are pairs (a, \bar{c}) such that $a \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $a < c_0$. The forcing order is $(b, \bar{d}) \geq (a, \bar{c})$ (recall the stronger condition is the larger one) iff $a \subseteq b$ and $b \setminus a$ is a union of finitely many of the c_n and \bar{d} is a condensation of \bar{c} .*

Definition 2.4. *Given a centred system $\mathcal{C} \subseteq (\mathbb{F})^\omega$, the notion of forcing $\mathbb{M}(\mathcal{C})$ consists of all pairs (s, \bar{a}') , such that $s \in \mathbb{F}$ and there is $\bar{a} \in \mathcal{C}$ such that \bar{a}' is an end-segment of \bar{a} , i.e., $\bar{a}' = \langle a_n : n \geq k \rangle$. The forcing order is the same as in the Matet forcing. In the special case that \mathcal{C} is the set of members of a \sqsubseteq^* -descending sequence \bar{a}^α , $\alpha < \beta$, we also write $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ for $\mathbb{M}(\mathcal{C})$.*

In this section we use $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ for \sqsubseteq^* -descending sequences of length 1, of length $< \kappa$ and of length κ where $\kappa = (2^\omega)^\mathbf{V}$ is assumed to be regular. The forcing $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ diagonalises (“shoots a real through”) $\bigcup \{a_n^\alpha : n < \omega\}$, $\alpha < \beta$. We write $\text{set}(\bar{a})$ for $\bigcup \{a_n : n < \omega\}$.

That means that in the generic extension $\mathbf{V}^{\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)}$ the real $s = \bigcup \{w : \exists \bar{a}(w, a) \in G\}$ is an almost subset of any $\text{set}(\bar{a}^\alpha)$. So, in $\mathbf{V}[G]$ the semi-filter generated by $\{\text{set}(\bar{a}^\alpha) : \alpha < \beta\}$ is meagre even if it was not meagre before. This means that our forcings are very specific: They destroy the non-meagreness of some sets, but may preserve the non-meagreness of others. We now work on this and a related preservation property.

We let $f : \omega \rightarrow \omega$ be strictly increasing with $f(0) = 0$ and let $x : \omega \rightarrow 2$. The meagre set coded by f and x is

$$(2.1) \quad M_{(f,x)} = \{y \in {}^\omega 2 : (\forall^\infty n)(y \upharpoonright [f(n), f(n+1)) \neq x \upharpoonright [f(n), f(n+1)))\}$$

see [2, 2.2.4]. Every meagre subset of ${}^\omega 2$ is a subset of a meagre set of the form $M_{(f,x)}$, which is F_σ .

We write names for reals and for meagre sets in c.c.c. forcings \mathbb{P} in a standardised form. Let $g : \omega \rightarrow H(\omega)$, $H(\omega)$ is the set of hereditarily finite sets. A standardised name for g is

$$\underline{g} = \text{Name}(\bar{k}, \bar{p}) = \{\langle (n, k_{n,m}), p_{n,m} \rangle : n, m \in \omega\},$$

such that $\{p_{n,m} : m \in \omega\}$ is predense in \mathbb{P} and $p_{n,m} \Vdash \underline{g} \upharpoonright n = \check{k}_{n,m}$, $k_{n,m} \in H(\omega)$, and such that $k_{n',m'} \upharpoonright n = k_{n,m}$ if $p_{n',m'}$ and $p_{n,m}$ are compatible and $n' \geq n$. In the case of meagre sets we let \underline{g} code a name (f, x) for a pair (f, x) as in Equation (2.1). This is done as follows: In order to ease notation, we let $k_{n,m}$ consist of $f_n : n + 2 \rightarrow \omega$ and $x_n : f_n(n + 1) \rightarrow 2$ for a strictly increasing $f_n \in {}^{n+2}\omega$ and $x_n \in {}^{f(n+1)}2$. Thus the $k_{n,m}$ are possibilities for $f \upharpoonright n + 2$, $x \upharpoonright f(n + 1)$.

If \underline{g} codes (f, x) we call the name given by entering (f, x) into Equation (2.1) simply $M_{\underline{g}}$. The code is evaluated in various ZFC models and always gives a meagre set.

We write $\mathbb{P} \subseteq_{ic} \mathbb{P}'$ iff $\mathbb{P} \subseteq \mathbb{P}'$ and for any $p, q \in \mathbb{P}$, if p and q are incompatible in \mathbb{P} then they are also incompatible in \mathbb{P}' . If $\mathbb{P} \subseteq_{ic} \mathbb{P}'$ then not every standardised \mathbb{P} -name for a real is also \mathbb{P}' -name for a real. This happens, however, if any maximal antichain $\{p_{n,m} : m \in \omega\}$ in \mathbb{P} stays maximal in \mathbb{P}' . This explains our adding the clauses “is a $\mathbb{M}(\bar{b}^\beta)$ -name of a meagre set that can be construed as an $\mathbb{M}(\bar{c}^\beta)$ -name” in the inductive construction. In the end we want to evaluate only names in the final order and each such name appears at some stage of countable cofinality (see Lemma 2.5) and then can be construed also as a name of a forcing order of any later stage.

We let $\mathbb{P} < \mathbb{Q}$ denote that \mathbb{P} is a complete suborder of \mathbb{Q} . If $\mathcal{C} \subseteq \mathcal{C}'$ are centred systems, then $\mathbb{M}(\mathcal{C}) \subseteq_{ic} \mathbb{M}(\mathcal{C}')$. The constructions in the section are based on the following fact: If \underline{g} is a $\mathbb{M}(\mathcal{C})$ -name and an $\mathbb{M}(\mathcal{C}')$ -name, $\mathcal{C} \subseteq \mathcal{C}'$ and $p \in \mathbb{M}(\mathcal{C})$ and $k \in H(\omega)$, then: Iff $p \Vdash_{\mathbb{M}(\mathcal{C})} \underline{g}(\check{n}) = \check{k}$ then $p \Vdash_{\mathbb{M}(\mathcal{C}')} \underline{g}(\check{n}) = \check{k}$. In general $\mathbb{M}(\mathcal{C})$ is not a complete suborder of $\mathbb{M}(\mathcal{C}')$, and in general the Cohen forcing, which is equivalent to $\mathbb{M}(\bar{a})$, is not a complete suborder of $\mathbb{M}(\mathcal{C})$. For example, there are $\mathbb{M}(\mathcal{C})$ that preserve an ultrafilter from the ground model [14, Theorem 2.5].

The following, until the end of this section, is used as a crucial step. We construct a σ -centred forcing $\mathbb{Q} = \mathbb{M}(\mathcal{C})$ as the union of an \subseteq_{ic} -increasing chain $\mathbb{M}(\mathcal{C}_\alpha)$, $\alpha < \kappa$. \mathbb{Q} will serve as an iterand in an iteration of length κ^+ with finite or with countable supports. If there is $\bar{d} \in \mathcal{C}$ such that $\forall \bar{c} \in \mathcal{C}$, $\bar{d} \sqsubseteq^* \bar{c}$, then, $\mathbb{M}(\mathcal{C})$ is equivalent to $\mathbb{M}(\bar{d})$ and this is in turn equivalent to Cohen forcing.

Lemma 2.5. *Let $\{\bar{c}^\varepsilon : \varepsilon < \delta\}$, be a \sqsubseteq^* -centred set. Assume $\mathbb{Q} = \mathbb{M}(\{\bar{c}^\varepsilon : \varepsilon < \delta\})$ and $\text{cf}(\delta) > \omega_0$ and \underline{g} is a \mathbb{Q} -name for a member of ${}^\omega H(\omega)$ or of a meagre set. Then we can find an $\varepsilon_0 < \delta$ such that for every $\varepsilon \in [\varepsilon_0, \delta)$ there are $p_{n,m} \in \mathbb{M}(\bar{c}^\varepsilon)$ and $k_{n,m} \in {}^\omega H(\omega)$ such that $\{p_{n,m} : m < \omega\}$ is predense in \mathbb{Q} and $p_{n,m} \Vdash_{\mathbb{Q}} \underline{g}(n) = k_{n,m}$.*

Proof. We assume that $\underline{g} = \{\langle (n, h_{n,m}), q_{n,m} \rangle : m, n < \omega\}$. Since $\text{cf}(\delta) > \omega$, there is some $\varepsilon_0 < \delta$ such that all $q_{n,m}$ are in $\mathbb{M}(\{\bar{c}^\beta : \beta \leq \varepsilon_0\})$. Now, given $\varepsilon \in [\varepsilon_0, \delta)$, we take

$$I_n = \{q \in \mathbb{M}(\bar{c}^\varepsilon) : (\exists m)(q \geq_{\mathbb{Q}} q_{n,m})\}.$$

Then I_n is predense in \mathbb{Q} . Now let $p_{n,m}$, $m < \omega$, list I_n and choose $k_{n,m}$ such that $p_{n,m} \Vdash_{\mathbb{Q}} \underline{g}(n) = k_{n,m}$. Then \bar{k}, \bar{p} describe \underline{g} as desired. \square

In this section we deal with the simple case that $\mathcal{C} = \{\bar{c}^\varepsilon : \varepsilon < \delta\}$ is the range of a \sqsubseteq^* -descending sequence $\langle \bar{c}^\varepsilon : \varepsilon < \delta \rangle$. The following lemma is the key to the preservation properties used in our construction. Therefore we prove it very much in detail. Recall $\text{MA}_{<\kappa}(\sigma\text{-centred})$ is Martin's axiom for σ -centred posets and $< \kappa$ dense sets. A poset \mathbb{P} is called centred if for any finite $F \subseteq \mathbb{P}$ there is q stronger than any of the $p \in F$. \mathbb{P} is σ -centred if it is the union of countably many centred sub-posets. Let Γ be a class of forcings. $\text{MA}_{<\kappa}(\Gamma)$ says: For any $\mathbb{P} \in \Gamma$ for any $\gamma < \kappa$ for any collection $\{D_\alpha : \alpha < \gamma\}$ there is a filter $G \subseteq \mathbb{P}$ such that $(\forall \alpha < \gamma)(D_\alpha \cap G \neq \emptyset)$.

In the next lemma we use a technique called “sealing antichains” or “processing names” that was used in the set theory of the reals [1, 11, 13, 15, 25] and possibly elsewhere and is a very important technique in constructing forcings under the assumption of large cardinals. “Sealing the antichains” is a crucial step in the constructions of the iterands $\mathbb{Q}_\alpha = \mathbb{M}(\langle \bar{c}_\varepsilon : \varepsilon < \omega_1 \rangle)$ in the proofs of Lemma 5.3 and of Lemma 5.4. It allows us to preserve the non-meagreness of up to $(2^\omega)^\mathbf{V}$ sets and to preserve the density in the \sqsubseteq^* -order. We state and prove the following lemma for an arbitrary regular κ , though we use it for $\kappa = \aleph_1$ in the proof of the main theorem.

Lemma 2.6. *Assume that κ is a regular uncountable cardinal, $2^\omega = \kappa$, $\text{MA}_{<\kappa}(\sigma\text{-centred})$, and that $\mathcal{X} = \{x_\alpha : \alpha < \kappa\}$ is not meagre. Then there is a \sqsubseteq^* -descending sequence $\langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle$ such that of size κ such that $\mathbb{Q} = \mathbb{M}(\bar{c}^\varepsilon : \varepsilon < \kappa)$ fulfils*

$$\Vdash_{\mathbb{Q}} \text{“}\mathcal{X} \text{ is not meagre”}$$

Proof. Let $\langle \bar{b}^\alpha, g^\alpha : \varepsilon < \kappa \rangle$ list the pairs (\bar{b}, g) such that $\bar{b} \in (\mathbb{F})^\omega$ and $g = \{ \langle (n, k_{n,m}), p_{n,m} \rangle : m, n \in \omega \}$ is an $\mathbb{M}(\bar{b})$ -name for a meagre set. We assume that each pair (\bar{b}, g) appears in the list κ many times. Let $g^\delta = \{ \langle (n, k_{n,m}^\delta), p_{n,m}^\delta \rangle : m, n \in \omega \}$.

We choose by induction on $\varepsilon < \kappa$ a sequence $\bar{c}^\varepsilon \in (\mathbb{F})^\omega$ with the following properties:

- (a) If $\delta < \varepsilon$ then $\bar{c}^\varepsilon \sqsubseteq^* \bar{c}^\delta$.
- (b) If $\varepsilon = \delta + 1$ and for some $\gamma \leq \delta$, $\bar{b}^\delta = \bar{c}^\gamma$ and g^δ is a $\mathbb{M}(\bar{b}^\delta)$ -name of a meagre set that can be construed as an $\mathbb{M}(\bar{c}^\delta)$ -name, then \bar{c}^ε guarantees that for some $\zeta_\varepsilon < \kappa$,

$$\Vdash_{\mathbb{Q}} x_{\zeta_\varepsilon} \notin M_{g^\delta}.$$

For $\varepsilon = 0$ we let $\bar{c}^0 = \langle \{n\} : n < \omega \rangle$.

Let $\varepsilon < \kappa$ be a limit ordinal. We apply $\text{MA}_{<\kappa}(\sigma\text{-centred})$ to the σ -centred forcing notion $\{(\bar{c}, n, F) : \bar{c} \text{ is a finite unmeshed sequence of subsets of } n \text{ and } F \text{ is a finite subset of } \varepsilon\}$, (Israeli) ordered by $(\bar{b}, n', F') \geq (\bar{c}, n, F)$ iff $n' \geq n$, $F' \supseteq F$, $\bar{b} \sqsubseteq \bar{c}$, $b_i = c_i$ for $i < n$ and $(\forall \gamma \in F)(\forall k)(b_k \subseteq [n, n'] \rightarrow b_k \in FU(\bar{c}'))$, and the dense sets $\mathcal{J}_{\delta, n} = \{(\bar{c}, m, F) : \bigcup \bar{c} \setminus n \neq \emptyset \wedge \delta \in F \wedge m \geq n\}$, $\delta < \varepsilon$, $n < \omega$, and thus we get a filter G intersecting all the $\mathcal{J}_{\delta, n}$ and set $\bar{c}^\varepsilon = \bigcup \{ \bar{c} : (\exists n, F)((\bar{c}, n, F) \in G) \}$. Then \bar{c}^ε is as desired.

Step $\varepsilon = \delta + 1$, and \bar{c}^δ is chosen. We assume that for some $\gamma \leq \delta$, $\bar{b}^\delta = \bar{c}^\gamma$ and g^δ is a $\mathbb{M}(\bar{b}^\delta)$ -name of a meagre set that has an equivalent $\mathbb{M}(\bar{c}^\delta)$ -name. Otherwise we can take $\bar{c}^{\delta+1} = \bar{c}^\delta$.

Recall that by our coding

$$g^\delta = \text{Name}(\bar{k}^\delta, \bar{p}^\delta) = \{ \langle (n, k_{n,m}^\delta), p_{n,m}^\delta \rangle : n, m \in \omega \}$$

and $k_{n,m}^\delta = (f_{n,m}^\delta, x_{n,m}^\delta)$, $f_{n,m}^\delta : n + 2 \rightarrow \omega$, $x_{n,m}^\delta : f_{n,m}^\delta(n + 1) \rightarrow 2$

By induction on $r \in \omega$ we first choose $c_r^{\varepsilon+} \in \mathbb{F}$ and $b(r) \in \omega$, $u_r \in \mathbb{F}$ such that $c_r^{\varepsilon+} = \bigcup \{ c_n^\delta : n \in u_r \}$. We let $c_0^{\varepsilon+} = c_0^\delta$, $b(0) = \max(c_0^{\varepsilon+}) + 1$, $u_0 = \{0\}$.

Suppose that $c_r^{\varepsilon+}$, $b(r)$ are chosen. Let $\{w_{r,i} : i < 2^{b(r)}\}$ enumerate all subsets of $b(r)$. Now by subinduction on $i < 2^{b(r)}$ we choose $n^\delta(r, i) = n^\delta(r, w_{r,i})$, $m^\delta(r, i) = m^\delta(r, w_{r,i})$ such that

- (1) $p_{n^\delta(r,0), m^\delta(r,0)}^\delta \geq (w_{r,0}, \bar{c}^\delta \upharpoonright [\max(u_r) + 1, \omega))$.
- (2) For $1 \leq i \leq 2^{b(r)} - 1$, $p_{n^\delta(r,i), m^\delta(r,i)}^\delta = (w_{r,i} \cup c(p_{n^\delta(r,i), m^\delta(r,i)}^\delta), \bar{c}((p_{n^\delta(r,i), m^\delta(r,i)}^\delta)))$.
- (3) For $0 \leq i \leq 2^{b(r)} - 1$, $\bar{c}((p_{n^\delta(r,i), m^\delta(r,i)}^\delta))$ is an end segment of \bar{c}^δ .
- (4) For $0 \leq i < 2^{b(r)} - 1$,

$$\begin{aligned} & (c(p_{n^\delta(r,i), m^\delta(r,i)}^\delta), \bar{c}(p_{n^\delta(r,i), m^\delta(r,i)}^\delta)) \\ & \leq (c(p_{n^\delta(r,i+1), m^\delta(r,i+1)}^\delta), \bar{c}(p_{n^\delta(r,i+1), m^\delta(r,i+1)}^\delta)). \end{aligned}$$

- (5) For $0 \leq i \leq 2^{b(r)} - 1$, $p_{n^\delta(r,i), m^\delta(r,i)}^\delta$ determines $g \upharpoonright n^\delta(r,i) + 2$, and in particular it determines

$$x_{n^\delta(r,i), m^\delta(r,i)}^\delta : f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i) + 1) \rightarrow 2.$$

- (6) For $0 \leq i < 2^{b(r)} - 1$, $n^\delta(r, w_{r,i}) \geq r$ and

$$b(r) \leq f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i)) < f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i) + 1) \leq f_{n^\delta(r,i+1), m^\delta(r,i+1)}^\delta(n^\delta(r,i+1)).$$

- (7) $B(r) = \bigcup \{ [f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i)), f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i)+1)) : i < 2^{b(r)} \}$.

Once the subinduction is performed, we choose a finite set

$$c_{r+1}^{\varepsilon+} = \bigcup \{ c(p_{n^\delta(r,i), m^\delta(r,i)}^\delta) : i < 2^{b(r)} \}$$

This also determines u_{r+1} . We let

$$b(r+1) = \max(c_{r+1}^{\varepsilon+}) + 1.$$

Thus the induction step is finished. We have

$$(2.2) \quad \bar{c}(p_{n^\delta(r, 2^{b(r)}-1), m^\delta(r, 2^{b(r)}-1)}^\delta) = \bar{c}^\delta \upharpoonright [\max(u_{r+1}) + 1, \infty).$$

Since \mathcal{X} is not meagre there is some $\zeta_\varepsilon < \kappa$ such that there is an infinite set $Y = \{r_i : i < \omega\} \subseteq \omega$ such that

- (m1) $\langle r_i : i < \omega \rangle$ is an increasing enumeration of Y , and
(m2) for every $j \in \omega$,

$$x_{\zeta_\varepsilon} \upharpoonright B(r_j) = \bigcup_{i < 2^{b(r_j)}} \left(x_{n^\delta(r_j,i), m^\delta(r_j,i)}^\delta \upharpoonright [f_{n^\delta(r_j,i), m^\delta(r_j,i)}^\delta(n^\delta(r_j,i)), f_{n^\delta(r_j,i), m^\delta(r_j,i)}^\delta(n^\delta(r_j,i) + 1)) \right).$$

We let

$$(2.3) \quad \bar{c}^\varepsilon = \langle c_{r_j+1}^{\varepsilon+} : j < \omega \rangle.$$

Now we show that $\mathbb{Q} = \mathbb{M}(\bar{c}^\varepsilon : \varepsilon < \kappa)$ is as desired. It is σ -centred, because for every $w \in \mathbb{F}$, $\mathbb{Q}_w = \{(w, \bar{c}^\delta \upharpoonright [\ell, \omega)) : \ell \in \omega, w < c_\ell^\delta, \delta \in \kappa\}$ is centred. We show that

$$\Vdash_{\mathbb{Q}} \mathcal{X} \text{ is not meagre.}$$

Assume towards a contradiction that there is a \mathbb{Q} -name \underline{g} for a meagre set and there is $p \in \mathbb{Q}$ such that $p \Vdash_{\mathbb{Q}} \mathcal{X} \subseteq M_{\underline{g}}$. Since $\text{cf}(\kappa) > \omega$, by Lemma 2.5 there is some $\gamma < \kappa$ such that \underline{g} is an $\mathbb{M}(\bar{c}^\gamma)$ -name. Since in the enumeration every name appears cofinally often, for some $\delta \geq \gamma$ we have $(\bar{b}^\delta, \underline{g}^\delta) = (\bar{c}^\gamma, \underline{g})$. So at stage $\varepsilon = \delta + 1$ in our construction we take care of \underline{g} 's equivalent $\mathbb{M}(\bar{c}^\delta)$ -name $\text{Name}(\bar{k}^\delta, \bar{p}^\delta)$. Let ζ_ε and \bar{c}^ε be as in this step.

Assume that there are some $q \geq p$ and some $n(*)$ such that $q \Vdash_{\mathbb{Q}} (\forall n \geq n(*))(x_g \upharpoonright [f_g(n), f_g(n+1)) \neq x_{\zeta_\varepsilon} \upharpoonright [f_g(n), f_g(n+1))$. By the form of \mathbb{Q} ,

$q = (s, \bar{c}^{\varepsilon(1)})$ for some $\varepsilon(1) \geq \varepsilon$ and some s , such that $\bar{c}^{\varepsilon(1)}$ is a condensation of \bar{c}^ε . So there are i, j, ℓ_j, ℓ_{j+1} and $r_i \geq n(*)$ with r_i as in (m1) for the construction step of \bar{c}^ε and with (m2) and with \bar{c}_ε according to Equation (2.3) such that $c_j^{\varepsilon(1)} \subseteq \ell_{j+1}$ and $c_j^{\varepsilon(1)} \cap [\ell_j, \ell_{j+1}) = c_i^\varepsilon$. However, $c_i^\varepsilon = c_{r_i+1}^{\varepsilon+}$. We let $s' = s \cup (\bigcup \bar{c}^{\varepsilon(1)} \cap [0, \ell_j))$, and we let $q' = (s' \cup c_i^\varepsilon, c_{j+1}^{\varepsilon(1)}, \dots)$.

Then q' witnesses that q and $p_{n^\delta(r_i, s'), m^\delta(r_i, s')}$ are compatible, because $q \leq q'$ and $p_{n^\delta(r_i, s'), m^\delta(r_i, s')} \leq q'$. Property (m2) in the choice of Y together with Equation 2.3 yield $q' \Vdash_{\mathbb{Q}} \dot{x} \upharpoonright [f^\delta(n^\delta(r_i, s')), f^\delta(n^\delta(r_i, s') + 1)) = x_{\zeta_\varepsilon} \upharpoonright [f^\delta(n^\delta(r_i, s')), f^\delta(n^\delta(r_i, s') + 1))$. Since $n^\delta(r_i, s') \geq r_i \geq n(*)$, this is a contradiction. \square

Theorem 2.7. *Let \mathbb{M} be the full Matet forcing and let \mathcal{X} be not meagre. Then in $\mathbf{V}^{\mathbb{M}}$, \mathcal{X} is not meagre.*

Proof. This is, after the proof of Lemma 2.6, a density argument in the \sqsubseteq^* -order: Given \mathcal{X} , an \mathbb{M} -name g then we do a similar construction, this time for all \bar{c} in g (these are \aleph_1 many). The sequences $\bar{c}(p_{n^\delta(r, i), m^\delta(r, i)})$ and $\bar{c}^{\varepsilon+}$ in Equation 2.2 are in general not end segments but the results of suitable condensations. \square

Together with [30, Theorem 61] this allows us to determine the value of the cardinal characteristics \mathfrak{s} and $\text{univ}(\mathcal{M})$, the smallest cardinality of a non-meagre set, in the so-called Matet model:

Corollary 2.8. *In the countable support iteration of \mathbb{M} of length ω_2 , $\text{univ}(\mathcal{M}) = \aleph_1$.*

Since $\mathfrak{s} \leq \text{univ}(\mathcal{M})$, also \mathfrak{s} is \aleph_1 in the Matet model.

Now we prove that density in the condensation order, a property closely related to non-meagreness, can be preserved.

Definition 2.9. *Let $\bar{a} \in (\mathbb{F})^\omega$. A set $D \subseteq (\mathbb{F})^\omega$ is called dense below \bar{a} (in the \sqsubseteq^* -order) if $(\forall \bar{b} \sqsubseteq^* \bar{a})(\exists \bar{c} \sqsubseteq^* \bar{b})(\bar{c} \in D)$.*

The following lemma about sealing and preserving density in the \sqsubseteq^* -order is crucial for the choice of the iterands of our forcing.

Lemma 2.10. *Assume that κ is a regular uncountable cardinal, $2^\omega = \kappa$, $\text{MA}_{<\kappa}(\sigma\text{-centred})$, and that D is dense below \bar{a} in the \sqsubseteq^* -order. Then there is a \sqsubseteq^* -descending sequence $\langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle$ such that of size κ such that $\mathbb{Q} = \mathbb{M}(\bar{c}^\varepsilon : \varepsilon < \kappa)$ fulfils*

$$\Vdash_{\mathbb{Q}} \text{“} D \text{ is dense below } \bar{a} \text{ in the } \sqsubseteq^* \text{-order ”}$$

Proof. The proof is similar Lemma 2.6. We indicate the change: We let $x_n: f_n(n+1) \rightarrow 2$ be such that x_n is not zero everywhere on $[f_n(n), f_n(n+1))$. Since $D = \{x_\zeta : \zeta < \kappa\}$ is dense there are $\zeta_\varepsilon < \kappa$ and $\bar{c} \in (\mathbb{F})^\omega$ such that (m1) holds and

(m2') For every $j \in \omega$,

$$x_{\zeta_\varepsilon} \restriction \bigcup \{B(r) : r \in e_j\} = \bigcup_{r \in e_j, i < 2^{b(r)}} \left(x_{n^\delta(r,i), m^\delta(r,i)} \restriction \left[f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i)), f_{n^\delta(r,i), m^\delta(r,i)}^\delta(n^\delta(r,i) + 1) \right) \right).$$

We let

$$\bar{c}^\varepsilon = \langle \bigcup \{c_{r+1}^{\varepsilon+} : r \in e_j\} : j < \omega \rangle.$$

Then the rest of the proof is again as in Lemma 2.6. \square

Corollary 2.11. (a) \mathbb{M} preserves every dense set.

(b) In the countable support iteration of \mathbb{M} , any dense set from the ground model is a dense set.

Proof. Statement (b) follows from properness and [2, Theorem 6.1.18]. \square

The discussion from here to the end of the section is a digression from the proof. Preserving non-meagreness is different from preserving density in \sqsubseteq^* : Kellner and Shelah give a non-meagre set and a countable support iteration such that each initial segment preserves the non-meagreness, but the countable support limit does not [18, Example 4.1]. The author thanks Dilip Raghavan for pointing this out to her. The density of a set in the \sqsubseteq^* -order is preserved in countable support limits by [2, Theorem 6.1.18].

Preserving the non-meagreness of a semifilter is, by Lemma 1.1 just preserving \leq^* -unboundedness. This is a weaker preservation property. An example is Blass–Shelah forcing that is almost ω^ω -bounding and hence preserves any \leq^* -unbounded family, and makes the ground model meagre by adding an unsplit real [9]. For semifilters there is a connection between non-meagreness and a reminiscent of the \sqsubseteq^* -order:

Proposition 2.12. ([7, Prop. 9.4]) *Let N be a semifilter. Then N is not meagre iff $(\forall \bar{a} \in (\mathbb{F})^\omega)(\exists A \in [\omega]^\omega)(\bigcup_{n \in A} a_n \subseteq N)$.*

Proof. Assume that N is not meagre. For $\bar{a} \in (\mathbb{F})^\omega$, let $M_{\bar{a}} = \{c \in [\omega]^\omega : (\neg \exists^\infty n)(a_n \supseteq c)\}$. The set $M_{\bar{a}}$ is meagre and hence there is $c \in N \setminus M_{\bar{a}}$. Now assume that N is meagre. By Talagrand's Lemma 1.1 there is a finite-to-one f such that $f(N)$ is the Fréchet filter. Let $\bar{a} = \langle [f(2n), f(2n+1)) : n \in \omega \rangle$. Then there is no union of infinitely many $[f(n), f(n+1))$ that is a member of N . \square

3. PRESERVING A P -POINT FROM THE GROUND MODEL

In this section we continue to consider preservation properties of the iterands \mathbb{Q}_α for $\alpha \in \omega_2$. We specialise the forcing posets $\mathbb{M}(\mathcal{C})$ further.

For this, we recall some properties of filters on the set \mathbb{F} of all nonempty finite subsets for ω . Again our nomenclature follows Blass [3] and Eisworth [14]. The results we recollect in this section are Hindman's and Eisworth's (see [14]).

Definition 3.1. *A non-principal filter \mathcal{F} on \mathbb{F} is said to be an ordered-union filter if it has a basis of sets of the form $\text{FU}(\bar{d})$ for $\bar{d} \in (\mathbb{F})^\omega$. Let μ be an uncountable cardinal. An ordered-union filter is said to be $< \mu$ -stable if, whenever it contains $\text{FU}(\bar{d}_\alpha)$ for $\bar{d}_\alpha \in (\mathbb{F})^\omega$, $\alpha < \kappa$, for some $\kappa < \mu$, then it also contains some $\text{FU}(\bar{e})$ for some \bar{e} that is almost a condensation of \bar{d}_α for $\alpha < \kappa$. For " $< \omega_1$ -stable" we say "stable".*

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q -points [3] and there are models without Q -points [29]. With the help of Hindman's theorem one shows that under $\text{MA}(\sigma\text{-centred})$ stable (even $< 2^\omega$ -stable) ordered-union ultrafilters exist [3]. We will construct suitable stable ordered-union ultrafilters for the choice of \mathbb{Q}_α , $\alpha \in \omega_2$, by induction on ω_1 using CH. We recall Hindman's theorem:

Theorem 3.2. (Hindman, [16, Corollary 3.3]) *If the set \mathbb{F} is partitioned into finitely many pieces then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that $\text{FU}(\bar{d})$ is included in one piece.*

The theorem also holds if instead of \mathbb{F} we partition only $\text{FU}(\bar{c})$ for some $\bar{c} \in (\mathbb{F})^\omega$, the homogeneous sequence \bar{d} given by the theorem is then a condensation of \bar{c} .

Corollary 3.3. *Under CH for every $\bar{a} \in (\mathbb{F})^\omega$ there is a stable ordered union ultrafilter \mathcal{U} such that $\text{FU}(\bar{a}) \in \mathcal{U}$.*

Definition 3.4. *Given an ordered-union ultrafilter \mathcal{U} on \mathbb{F} we let $\mathbb{M}_{\mathcal{U}}$ consist of all pairs $(s, \bar{c}) \in \mathbb{M}$, such that $s \in \mathbb{F}$ and $\text{FU}(\bar{c}) \in \mathcal{U}$. The forcing order is the same as in the Matet forcing.*

It is well known [23, 4] that Matet forcing \mathbb{M} can be decomposed into two steps $\mathbb{M} = \mathbb{P} * \mathbb{M}_{\mathcal{U}}$, such that $\mathbb{P} = ((\mathbb{F})^\omega, \sqsupseteq^*)$ is $< \omega_1$ -closed (that is, every descending sequence of conditions of countable length has a lower bound) and adds a stable ordered-union ultrafilter \mathcal{U} on the set \mathbb{F} . In particular \mathbb{M} is proper.

In order to state a preservation property of $\mathbb{M}(\mathcal{U})$, we need the following definition.

Definition 3.5. *Let \mathcal{U} be a filter on \mathbb{F} . The core of \mathcal{U} is the filter $\Phi(\mathcal{U})$ such that*

$$X \in \Phi(\mathcal{U}) \text{ iff } (\exists \text{FU}(\bar{c}) \in \mathcal{U})(\bigcup_{n \in \omega} c_n \subseteq X).$$

If \mathcal{U} is ultra on \mathbb{F} , then $\Phi(\mathcal{U})$ is not diagonalised (see [14, Prop. 2.3]) and also all finite-to-one images of $\Phi(\mathcal{U})$ are not diagonalised (same proof).

So $\Phi(\mathcal{U})$ is not meagre. $\Phi(\mathcal{U})$, though, is not ultra by finite to one. This is proved in [6]. The reason is: There are two ultrafilters $\min(\mathcal{U}) = \{\{\min(d) : d \in D\} : D \in \mathcal{U}\}$, $\max(\mathcal{U}) \supset \Phi(\mathcal{U})$ that are not nearly coherent.

The *Rudin-Blass ordering* on semifilters¹ is defined as follows: Let $\mathcal{F} \leq_{RB} \mathcal{G}$ iff there is a finite-to-one f such that $f(\mathcal{F}) \subseteq f(\mathcal{G})$. The following property of stable ordered-union ultrafilters \mathcal{U} will be important for our proof:

Theorem 3.6. (*Eisworth* [14, “ \rightarrow ” Theorem 4, “ \leftarrow ” Cor. 2.5, this direction works also with non- P ultrafilters]) *Let \mathcal{U} be a stable ordered-union ultrafilter on \mathbb{F} and let \mathcal{V} be a P -point. Iff $\mathcal{V} \not\leq_{RB} \Phi(\mathcal{U})$, then \mathcal{V} continues to generate an ultrafilter after we force with $\mathbb{M}_{\mathcal{U}}$.*

In the decomposition $\mathbb{M} = \mathbb{P} * \mathbb{M}_{\mathcal{U}}$, the stable ordered-union ultrafilter \mathcal{U} in the intermediate model $\mathbf{V}^{\mathbb{P}}$ fulfils $\Phi(\mathcal{U}) \not\leq_{RB} \mathcal{V}$ for any P -point \mathcal{V} in the ground model, and hence by Theorem 3.6, \mathbb{M} preserves P -points.

In Section 5 we work with a \sqsubseteq^* -descending sequence \bar{c}_ε , $\varepsilon < \omega_1$, with the property that $\text{FU}(\bar{c}_\varepsilon)$, $\varepsilon < \omega_1$, generates an ultrafilter \mathcal{U} on \mathbb{F} . Then this is a stable ordered-union ultrafilter and $\mathbb{M}_{\mathcal{U}} = \mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$.

4. FINITE-TO-FINITE RELATIONS

In this section we consider finite-to-one functions and their inverse relations. This will be used to handle the quantifier $(\forall R \in \mathbf{V}_\delta)$ in (I5).

Definition 4.1.

- (1) $\mathcal{R}^* = \{R \subseteq \omega \times \omega : (\forall m)(\exists^{<\aleph_0} n)(mRn) \wedge (\forall n)(\exists^{<\aleph_0} m)(mRn)\}$. The quantifier $\exists^{<\aleph_0}$ means that there are finitely many and at least one. We let the letter R range over elements of \mathcal{R}^* .
- (2) For $R, S \in \mathcal{R}^*$ we let $R^{-1} = \{(n, m) : (m, n) \in R\}$ and we let $R \circ S = \{(m, r) : (\exists n)((m, n) \in R \wedge (n, r) \in S)\}$. Note that the order is different from the one known in the composition of functions: We first “map” with R then with S .
- (3) For $a \subseteq \omega$, $R \in \mathcal{R}^*$ we let $R(a) = Ra = \{n : mRn, m \in a\}$.
- (4) For $\bar{c} = \langle c_n : n \in \omega \rangle \in (\mathbb{F})^\omega$, $R \in \mathcal{R}^*$ we let $R(\bar{c}) = \langle Rc_n : n \in \omega \rangle$. This can be meshed or even be not pairwise disjoint (that is not unmeshed), but it does not matter.

The purpose of $R \in \mathcal{R}^*$ is to increase infinite sets in a gentle manner, as with finite-to-one functions: Iff $f''x \subseteq f''y$, then $x \subseteq Ry$ for $R = \{(m, n) : f(n) = f(m)\}$. Another use is: For a finite-to-one f , $f(\mathcal{F}) = \{X : f^{-1}X \in \mathcal{F}\} = \{X : R(X) \in \mathcal{F}\}$, where xRy iff $f(y) = x$. Since f is a finite-to-one function, we have $R \in \mathcal{R}$. Iff for every $R \in \mathcal{R}^*$ there is some $R(X) \in \mathcal{F}$ such that $R(X) \notin \mathcal{V}$ then \mathcal{F} is not Rudin-Blass below \mathcal{V} . We shall use the “if”-direction of this criterion for $\mathcal{F} = \Phi(\mathcal{U})$ and $\mathcal{V} = \mathcal{E}$ for the choice

¹Usually only filters are considered. However, we explicitly want to use this order also for semifilters.

of the iterands. For any two sequences \bar{c}, \bar{d} in $(\mathbb{F})^\omega$ there is $R_{\bar{c}, \bar{d}}$ such that $R_{\bar{c}, \bar{d}}(\bar{c}) = \bar{d}$. Indeed, there are many such R , but we fix just one: $R_{\bar{c}, \bar{d}} = \bigcup \{c_n \times d_n : n \in \omega\}$.

Definition 4.2. Let $\bar{c} \in (\mathbb{F})^\omega$ and $s \subseteq \omega$. We denote by $\bar{c} \cap s$ the subsequence of the c_n such that $c_n \subseteq s$.

In the cases we use $\bar{c} \cap s$ it will be again an element of $(\mathbb{F})^\omega$.

5. ITERATED FORCING

We start with a ground model \mathbf{V} that fulfils CH and $\diamond(S_1^2)$ (and hence $2^{\aleph_1} = \aleph_2$).

In a countable support iteration of proper forcings of iterands size $\leq \aleph_1$ each real appears in a \mathbf{V}_α for some α with countable cofinality [31, Ch. III]. Recall our notation $\mathbf{V}_\alpha = \mathbf{V}^{\mathbb{P}_\alpha}$. A reflection property ensures that each non-meagre filter \mathcal{F} in the final model has ω_1 -club many $\alpha \in \omega_2$ such that $\mathcal{F} \cap \mathbf{V}_\alpha$ has a \mathbb{P}_α -name and is a non-meagre filter in \mathbf{V}_α (see [9, Item 5.6 and Lemma 5.10]). A subset of ω_2 is called ω_1 -club if it is unbounded in ω_2 and closed under suprema of strictly ascending sequences of lengths ω_1 . A subset of ω_2 is called ω_1 -stationary if it has non-empty intersection with every ω_1 -club. By well-known techniques based on coding \mathbb{P}_α -names for filters as subsets of ω_2 (e.g., such a coding is carried out in [28, Claim 2.8]) and based on the maximal principle (see, e.g., [20, Theorem 8.2]) the $\diamond(S_1^2)$ -sequence $\langle \mathcal{S}_\alpha : \alpha \in S_1^2 \rangle$ gives ω_1 -club often a \mathbb{P}_α -name \mathcal{S}_α for a non-meagre filter in \mathbf{V}_α such that for any non-meagre filter $\mathcal{F} \in V^{\mathbb{P}_{\omega_2}}$ there are ω_1 -stationarily many $\alpha \in S_1^2$ with $\mathcal{F} \cap V^{\mathbb{P}_\alpha} = \mathcal{S}_\alpha$. For names \dot{x} and objects x we use the rule $\dot{x}[G] = x$. Often we write x for \dot{x} , in particular for the generic reals s_α and the ultrafilters \mathcal{U}_α of our iterated forcing: the list of properties below is mostly about names.

For a $\mathbb{Q}_\alpha = \mathbb{M}_{\mathcal{U}_\alpha}$ -generic filter G^α over \mathbf{V}_α we let

$$s_\alpha = \bigcup \{c : \exists \bar{a} \in \mathcal{U}_\alpha(c, \bar{a}) \in G^\alpha\}.$$

This infinite subset s of ω is called the Matet-generic real and its $\mathbb{P}_{\alpha+1}$ -name is called \mathcal{s}_α or just s_α . The step function defined from the inverse of the increasing enumeration of this set s_α is called f_α and will be called “the generic finite-to-one function”. For $\mathcal{X} \subseteq [\omega]^\omega$, we let $\text{cl}(\mathcal{X}) = \{Y : (\exists X \in \mathcal{X})(Y \supseteq X)\}$.

We fix a diamond sequence $\langle \mathcal{S}_\alpha : \alpha \in S_1^2 \rangle$. We also fix a P -point $\mathcal{E} \in \mathbf{V}$ that will be preserved throughout our iteration. We fix an enumeration $\langle E_\varepsilon : \varepsilon < \omega_1 \rangle$ of a basis of \mathcal{E} such that each elements appears cofinally often. We let R be used for elements of \mathcal{R}^* , and $R \in \mathbf{V}_\alpha$ means $R \in \mathcal{R}^* \cap \mathbf{V}_\alpha$. We use $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$ for $(\mathbb{F})^\omega$, and $\bar{a} \in \mathbf{V}_\alpha$ means $\bar{a} \in (\mathbb{F})^\omega \cap \mathbf{V}_\alpha$.

We construct by induction on $\alpha \leq \omega_2$ a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ such that for any $\alpha \leq \omega_2$, the initial segment $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils:

- (I1) For all $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta}$ “ \mathbb{Q}_β is proper and $|\mathbb{Q}_\beta| \leq \aleph_1$ ”.
- (I2) $\Vdash_{\mathbb{P}_\alpha}$ “ $\text{cl}(\mathcal{E})$ is ultra”.
- (I3) If $\beta \in S_1^2 \cap \alpha$ and if S_β is a \mathbb{P}_α -name \mathcal{F} for a non-meagre filter in $\mathbf{V}^{\mathbb{P}_\beta}$, then $\Vdash_{\mathbb{P}_{\beta+1}}$ “ $f_\beta(\mathcal{F}) = f_\beta(\text{cl}(\mathcal{E}))$ ”. Recall, f_β is defined from s_β as above.
- (I4) We define for $\beta \in \omega_2$, $\bar{a} \in \mathbf{V}_\beta$

$$(5.1) \quad \begin{aligned} D_\beta := & \{ \bar{d} \in (\mathbb{F})^\omega \cap \mathbf{V}_\beta : \\ & (\forall \bar{b} \sqsubseteq^* \bar{d})(\exists \bar{e} \sqsubseteq^* \bar{b})(\forall i \in \beta \setminus S_1^2) \\ & (\forall R \in \mathcal{R}^* \cap \mathbf{V}_i)(\exists^\infty n)(e_n \not\subseteq R s_i) \}. \end{aligned}$$

Note that D_β is not absolute. We require: For all $\beta < \gamma$, $\beta, \gamma \in (\alpha + 1) \setminus (S_1^2 \cup \{\omega_2\})$:

$$(5.2) \quad (\forall \bar{a} \in \mathbf{V}_\beta) \left((\mathbf{V}_\beta \models \bar{a} \in D_\beta) \rightarrow (\mathbf{V}_\gamma \models \bar{a} \in D_\beta \wedge (\exists \bar{b} \sqsubseteq^* \bar{a})(\bar{b} \in D_\gamma)) \right).$$

- (I5) For all $\gamma \in \alpha \setminus S_1^2$: s_γ is \leq^* -unbounded over \mathbf{V}_γ and $(\forall \beta \in \gamma \setminus S_1^2)(\forall R \in \mathcal{R}^* \cap \mathbf{V}_\beta)(\mathbb{P}_{\gamma+1} \Vdash (s_\gamma \not\subseteq^* R s_\beta))$.

We first show that the existence of such an iteration implies our main theorem:

Lemma 5.1. *Assume that \mathbb{P} has the properties listed above. Then in \mathbf{V}_{ω_2} the filter dichotomy holds by (I3) and the semifilter*

$$\mathcal{S} = \{x \in [\omega]^\omega : (\exists \alpha \in \omega_2 \setminus S_1^2)(x \supseteq^* s_\alpha)\}$$

is not-meagre, not comeagre, and not ultra by finite-to-one.

Proof. By properness our iteration preserves \aleph_1 . It preserves \aleph_2 , because any collapse would appear at some intermediate step \mathbb{P}_α , but \mathbb{P}_α has size \aleph_1 and the \aleph_2 -c.c. So \aleph_1 and \aleph_2 are in the following used in \mathbf{V} and in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ and there is no danger.

By Talagrand’s lemma 1.1 and since the enumerating function of the s_α , $\alpha \in \omega_2 \setminus S_1^2$ form an \leq^* -unbounded family, the semifilter is not meagre.

Since FD implies NCF, the statement “ \mathcal{S} is not ultra by finite-to-one” is equivalent to “ \mathcal{S} is not nearly coherent with \mathcal{E} ”. Assume for a contradiction that \mathcal{S} is nearly coherent with \mathcal{E} . Then a finite-to-one function f with $f(\mathcal{E}) = f(\mathcal{S})$ would appear in some \mathbf{V}_α , $\text{cf}(\alpha) = \omega$ and $\alpha < \omega_2$. We take $\beta > \alpha$, $\beta \in \omega_2 \setminus S_1^2$. By the properties of \mathbb{Q}_β , the increasing enumeration of $f''s_\beta$ is \leq^* -unbounded over \mathbf{V}_α . Hence $f''s_\beta \not\subseteq^* f''E$ for any $E \in \mathcal{E}$. So $f(\mathcal{S}) \neq f(\mathcal{E})$.

Suppose that \mathcal{S} is comeagre. Then by Talagrand’s lemma, applied to $\{\omega \setminus X : X \notin \mathcal{S}\}$, there is $f \in \mathbf{V}_{\omega_2}$, f finite-to-one, such that $f(\mathcal{S})$ is dense in $([\omega]^\omega, \subseteq^*)$. There is $\alpha \in \omega_2 \setminus S_1^2$ such that such an $f \in \mathbf{V}_\alpha$. Then by (I5) for $\beta > \alpha$, $\beta \in \omega_1 \setminus S_1^2$, $f''s_\beta \not\subseteq^* s_\alpha$. However, $([s_\alpha]^\omega, \subseteq^*)$ has

$2^\omega = \aleph_2$ many almost disjoint subsets, and hence $\{f''s_\beta : \beta \in (\alpha+1) \setminus S_1^2\}$, being of size at most \aleph_1 , is not dense in $([s_\alpha]^\omega, \subseteq^*)$. So the whole set $f(\mathcal{S}) = \{f''s_\beta : \beta \in \omega_2 \setminus S_1^2\}$ is not \subseteq^* -dense below s_α . \square

We will use throughout the iteration that desirable properties of elements of $(\mathbb{F})^\omega$ are carried to later stages of the iteration:

- Fact 5.2.** (a) *If $(\forall i \in \beta \setminus S_1^2)(\forall R \in \mathbf{V}_i)(\exists^\infty n)(a_n \not\subseteq Rs_i)$ and $\bar{b} \sqsupseteq^* \bar{a}$, then $(\forall i \in \beta \setminus S_1^2)(\forall R \in \mathbf{V}_i)(\exists^\infty n)(b_n \not\subseteq Rs_i)$.*
- (b) *If $\mathbf{V}_\beta \models \bar{a} \in D_\beta$ and $\mathbf{V}' \supseteq \mathbf{V}_\beta$ and \mathbf{V}' preserves the density of D_β below \bar{a} then $\mathbf{V}' \models \bar{a} \in D_\beta$.*

Proof. The statement

$$(\forall i \in \beta \setminus S_1^2)(\forall R \in \mathbf{V}_i)(\exists^\infty n)(a_n \not\subseteq Rs_i)$$

is absolute for $\mathbf{V}' \supseteq \mathbf{V}_\beta$. \square

Now we prove by induction on $\alpha \leq \omega_2$ that such an iteration exists. The following lemma is for the successor steps $\alpha \mapsto \alpha + 1$ for $\alpha \in S_1^2$.

Lemma 5.3. *Assume that $\alpha \in S_1^2$ and that $\langle \mathbb{P}_\gamma, \mathbb{Q}_\delta : \gamma \leq \alpha, \delta < \alpha \rangle$ is defined with the properties (I1) to (I5). Then there is a σ -centred \mathbb{Q}_α such that $\langle \mathbb{P}_\gamma, \mathbb{Q}_\delta : \gamma \leq \alpha + 1, \delta < \alpha + 1 \rangle$ has properties (I1) to (I5).*

Proof. Let $G_\alpha \subseteq \mathbb{P}_\alpha$ be generic over \mathbf{V} and let $G_\beta = \mathbb{P}_\beta \cap G_\alpha$ for $\beta < \alpha$. Let $\langle \alpha_\varepsilon : \varepsilon < \omega_1 \rangle \in \mathbf{V}$ be continuously increasing with limit α , and each α_ε has cofinality ω_0 for $1 \leq \varepsilon < \omega_1$ and let $\alpha_0 = 0$.

We fix a sequence $\langle (\xi_\varepsilon, \zeta_\varepsilon, \bar{b}_\varepsilon, \bar{d}_\varepsilon, R_\varepsilon, B_\varepsilon, \bar{e}_\varepsilon) : \varepsilon < \omega_1 \rangle$, such that

- (a) for every $\zeta < \omega_1$, the sequence $\langle (\xi_\varepsilon, \zeta_\varepsilon, \bar{b}_\varepsilon, \bar{d}_\varepsilon, R_\varepsilon, B_\varepsilon, \bar{e}_\varepsilon) : \varepsilon < \zeta \rangle$ belongs to $\mathbf{V}_{\alpha_\zeta}$, $\xi_\varepsilon < \alpha_\varepsilon$, $\zeta_\varepsilon \leq \alpha_\varepsilon$,
- (b) $R_\varepsilon \in \mathcal{R}^*$,
- (c) $\bar{b}_\varepsilon, \bar{d}_\varepsilon \in \mathbf{V}_{\alpha_\varepsilon}$,
- (d) $B_\varepsilon \subseteq \mathbb{F}$,
- (e) \bar{e}_ε is a $\mathbb{M}(\bar{b}_\varepsilon)$ -name for a member of $(\mathbb{F})^\omega$,
- (f) every such tuple $(\xi_\varepsilon, \zeta_\varepsilon, \bar{b}_\varepsilon, \bar{d}_\varepsilon, R_\varepsilon, B_\varepsilon, \bar{e}_\varepsilon)$ appears in the sequence \aleph_1 times.

We now choose \bar{c}_ε by induction on $\varepsilon < \omega_1$ such that

- (α) $\bar{c}_0 = \langle \{i\} : i \in \omega \rangle$. If $\zeta < \varepsilon$ then $\bar{c}_\zeta \sqsupseteq^* \bar{c}_\varepsilon$.
- (β) (The forcing tasks) $\bar{c}_\varepsilon \in D_{\alpha_\varepsilon}$ and

$$(\forall^\infty k)(c_{\bar{c}_\varepsilon+1,k} \not\subseteq R_\varepsilon s_{\xi_\varepsilon}).$$

- (γ) (The Eisworth tasks) $\omega \setminus R_\varepsilon(\text{set}(\bar{c}_{\varepsilon+1})) \in R_\varepsilon(\text{cl}^{\mathbf{V}_{\alpha_\varepsilon+1}}(\mathcal{C}))$.
- (δ) (The Hindman tasks) $\text{FU}(\bar{c}_{\varepsilon+1})$ is included in B_ε or disjoint from B_ε .

- (ε) (The Blass–Laflamme tasks) $\bar{c}_{\varepsilon+1} \Vdash \check{f}_\alpha'' E_\varepsilon \in \check{f}_\alpha(\mathcal{F})$, if possible. \mathcal{F} is the current task, a non-meagre filter handed down by the diamond. This is derived as follows: Since \mathcal{F} is not meagre, the set

$$(5.3) \quad \mathcal{G}_1(E_\varepsilon, \mathcal{F}) = \{Z : \exists Y \in \mathcal{F} (\forall a, b \in Z) (a < b \rightarrow ([a, b] \cap Y \neq \emptyset \rightarrow [a, b] \cap E_\varepsilon \neq \emptyset))\}$$

is groupwise dense. For details see [7, Section 9]. So $(\forall \bar{c} \in (\mathbb{F})^\omega \cap \mathbf{V}_\alpha)(\exists \bar{b} \sqsubseteq^* \bar{c})(\text{set}(\bar{b}) \in \mathcal{G}_1(E_\varepsilon, \mathcal{F}))$. If there is such a $\bar{b} \in \mathbf{V}_{\alpha_{\varepsilon+1}}$, then we let $\bar{c}_{\varepsilon+1} = \bar{b}$. Otherwise we let $\bar{c}_{\varepsilon+1} = \bar{c}_\varepsilon$. This is Blass’ and Laflamme’s proof of the filter dichotomy [8]: $\bar{c}_{\varepsilon+1}$ ensures that $\mathbb{P}_{\alpha+1} \Vdash \check{f}_\alpha'' E_\varepsilon = \check{f}_\alpha'' Y \in \check{f}_\alpha(\mathcal{F})$ for a witness Y as in (5.3). Since $\mathcal{F} \in \mathbf{V}_\alpha$, this task can be fulfilled at some step ε , namely when there is $\bar{b} \sqsubseteq^* \bar{c}_\varepsilon$, $\bar{b} \in \mathcal{G}_1(E_\varepsilon, \mathcal{F})$, $\bar{b} \in \mathbf{V}_{\alpha_{\varepsilon+1}}$. Since every element of $\{E_\delta : \delta < \omega_1\}$ appears at cofinally many stages, fulfilling the task for a given E_δ will be possible at some stage when $E_\delta = E_\varepsilon$ is named in requirement (ε) .

- (ζ) (The sealing tasks) If \bar{e}_ε is an $\mathbb{M}(\bar{c}_\varepsilon)$ -name, we choose $\bar{c}_{\varepsilon+1}$ to secure: If D_{ζ_ε} is dense below \bar{d}_ε in $\mathbf{V}_{\alpha_\varepsilon}$, and if $\bar{e}_\varepsilon \in \mathbb{M}(\bar{c}_\varepsilon)$ stays a \mathbb{Q}_α -name, then \bar{e}_ε is not witnessing that D_{ζ_ε} is not dense below \bar{d}_ε in $\mathbf{V}_{\alpha+1}$, that is, the choice of $\bar{c}_{\varepsilon+1}$ according to (m2’) in the proof of Lemma 2.10 guarantees

$$\begin{aligned} \mathbf{V}_{\alpha+1} \models & \left(\bar{e}_\varepsilon \not\sqsubseteq^* \bar{d}_\varepsilon \right. \\ & \vee (\exists \bar{e} \sqsubseteq^* \bar{e}_\varepsilon)(\bar{e} \in \mathbf{V}_\alpha \wedge (\forall i \in \zeta_\varepsilon \setminus S_1^2) \\ & \left. (\forall R \in \mathcal{R}^* \cap \mathbf{V}_i)(\exists^\infty n)(e_n \not\sqsubseteq R s_i)) \right). \end{aligned}$$

We start the induction with $\alpha_0 = 0$, and we take an arbitrary $\bar{c}_0 \in (\mathbb{F})^\omega \cap \mathbf{V}_0$.

At limit steps ε we take the $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\zeta$ for all $\zeta < \varepsilon$. Since for every ζ , the set $\bar{c}_\varepsilon \in D_{\alpha_\zeta}$, we get $\bar{c}_\varepsilon \in D_{\alpha_\varepsilon}$.

We carry out the successor step, $\varepsilon = \delta + 1$. Suppose \bar{c}_δ is given. We work until further notice in $\mathbf{V}_{\alpha_\delta}$. We strengthen \bar{c}_δ five times in order to fulfil the current instance of the forcing task (β), the Eisworth task (γ), the Hindman task (δ), the Blass–Laflamme task (ε) (we already explained there how to fulfil these tasks), and the sealing task (ζ) (as explained in the proof of Lemma 2.10), and we call the outcome $\bar{c}_\delta^+ \sqsubseteq \bar{c}_\delta$. The names $R_\delta, \bar{d}_\delta, \bar{b}_\delta, B_\delta$ and \bar{e}_ε and the handed down names for members of \mathcal{F} are elements of $\mathbf{V}_{\alpha_\delta}$ and all the strengthening is done in $\mathbf{V}_{\alpha_\delta}$. Now we possibly leave $\mathbf{V}_{\alpha_\delta}$: We strengthen \bar{c}_δ^+ finally according to the induction hypothesis (I4): We step up from α_δ to α_ε : In $\mathbf{V}_{\alpha_\varepsilon}$, there is $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\delta^+$ with $\bar{c}_\varepsilon \in D_{\alpha_\varepsilon}$.

We let $\mathcal{U}_\alpha = \text{cl}(\{\text{FU}(\bar{c}_\varepsilon) : \varepsilon < \omega_1\})$. It is a stable ordered-union ultrafilter by (α) and (δ) . Then we take $\mathbb{Q}_\alpha = \mathbb{M}_{\mathcal{U}_\alpha}$. It is σ -centred and hence proper. So (I1) holds.

In \mathbf{V}_α , the P -point \mathcal{E} and is not Rudin-Blass above $\Phi(\mathcal{U}_\alpha)$. This is secured by (γ) , since all Rudin-Blass finite-to-one maps in \mathbf{V}_α are covered by the enumeration $\{R_\varepsilon : \varepsilon \in \omega_1\} = (\mathcal{R}^*)^{\mathbf{V}_\alpha}$. By Eisworth's Theorem 3.6, the successor \mathbb{Q}_α preserves “ $\text{cl}(\mathcal{E})$ is an ultrafilter”. So (I2) holds also for $\mathbb{P}_{\alpha+1}$. Item (I3) for $\alpha \in S_1^2$ is proved as in work by Blass and Laflamme [8], we use just the density in \sqsubseteq^* of the groupwise dense ideal $\mathcal{G}_1(E_\varepsilon, \mathcal{F})$.

Next we prove (I4) in the new cases, that is for some $\beta \in (\alpha+1) \setminus S_1^2 = \alpha \setminus S_1^2$ and for $\gamma = \alpha+1$ itself. The first part of (I4) follows from the sealing tasks. For the second part of (I4), we assume $\beta < \alpha$ and $\bar{d} \in \mathbf{V}_\beta$ such such that $\bar{d} \in D_\beta$ is given. Again, since every real appears at an iteration step of countably cofinality, there is ε such that $\beta \leq \alpha_\varepsilon$. By induction hypothesis there is $\bar{d}' \sqsubseteq \bar{d}$, $\bar{d}' \in D_{\alpha_\varepsilon}$ in $\mathbf{V}_{\alpha_\varepsilon}$. Then also $R_{\bar{d}', \bar{c}_\varepsilon} \in V_{\alpha_\varepsilon}$. So $\bar{d}' = R_{\bar{c}_\varepsilon, \bar{d}'} \bar{c}_\varepsilon \in D_{\alpha_\varepsilon}$ in $\mathbf{V}_{\alpha_\varepsilon}$. A density argument in $\leq_{\mathbb{Q}_\alpha}$ shows

$$(5.4) \quad (\forall \varepsilon < \omega_1)(\mathbb{Q}_\alpha \Vdash \text{“}s_\alpha \text{ is a union of infinitely many blocks of } \bar{c}_\varepsilon \text{ and a finite rest.”})$$

Hence $\Vdash_{\mathbb{P}_{\alpha+1}} \bar{c}_\varepsilon \cap s_\alpha \sqsubseteq^* \bar{c}_\varepsilon$. We have in $\mathbf{V}_{\alpha+1}$, $R_{\bar{d}', \bar{c}_\varepsilon}(\bar{d}' \cap R_{\bar{c}_\varepsilon, \bar{d}'}(s_\alpha)) =^* \bar{c}_\varepsilon \cap s_\alpha$, since s_α splits only finitely many $c_{\varepsilon, n}$. So in $\mathbf{V}_{\alpha+1}$,

$$\bar{d}'' := R_{\bar{c}_\varepsilon, \bar{d}'}(\bar{c}_\varepsilon \cap s_\alpha) = \bar{d}' \cap R_{\bar{c}_\varepsilon, \bar{d}'}(s_\alpha) \sqsubseteq \bar{d}'.$$

We claim that

$$(5.5) \quad \Vdash_{\mathbb{P}_{\alpha+1}} \bar{d}'' \in D_{\alpha+1}.$$

For this we show first

$$(5.6) \quad (\forall i \in (\alpha+1) \setminus S_1^2)(\forall R \in \mathcal{R}^* \cap \mathbf{V}_i)(\exists^\infty n)(d''_n \not\subseteq Rs_i).$$

The instances of Equation (5.6) for $i \in \alpha_\varepsilon \setminus S_1^2$ follow from $\bar{d}' \in D_{\alpha_\varepsilon}$ and the fact that the D_{α_ε} are dense also in $\mathbf{V}_{\alpha+1}$. The latter is secured by the sealing tasks.

The instances of Equation (5.6) for $i \geq \alpha_\varepsilon$ are seen as follows: Equation (5.4) implies together with the forcing tasks (β)

$$\begin{aligned} & (\forall i \in (\alpha+1) \setminus (S_1^2 \cup \alpha_\varepsilon))(\forall R \in \mathcal{R}^* \cap \mathbf{V}_i) \\ & ((\forall^\infty n)(c_{\varepsilon, n} \cap s_\alpha = \emptyset \vee c_{\varepsilon, n} \cap s_\alpha \not\subseteq Rs_i) \\ & \quad \wedge (\exists^\infty n)(c_{\varepsilon, n} \cap s_\alpha \neq \emptyset)). \end{aligned}$$

Since $R_{\bar{d}', \bar{c}_\varepsilon} \in \mathbf{V}_{\alpha_\varepsilon}$ we can read the quantifiers “ $(\forall R \in \mathcal{R}^* \cap \mathbf{V}_i)$ ” as follows: $(\forall i \in (\alpha+1) \setminus (S_1^2 \cup \alpha_\varepsilon))(\forall R \in \mathcal{R}^* \cap \mathbf{V}_i)(\exists^\infty n)(c_{\varepsilon, n} \cap s_\alpha \not\subseteq R_{\bar{d}', \bar{c}_\varepsilon} Rs_i)$. The non-inclusion $c_{\varepsilon, n} \cap s_\alpha \not\subseteq R_{\bar{d}', \bar{c}_\varepsilon}(Rs_i)$ implies $R_{\bar{c}_\varepsilon, \bar{d}'}(c_{\varepsilon, n} \cap s_\alpha) \not\subseteq R_{\bar{c}_\varepsilon, \bar{d}'} R_{\bar{d}', \bar{c}_\varepsilon} Rs_i$, so $d''_n \not\subseteq Rs_i$. So all instances of Equation (5.6) are shown. Now we take $\bar{d}''' \sqsubseteq^* \bar{d}'$, $\bar{d}''' \in D_{\alpha_\varepsilon}$ and start the proof anew with this instead of \bar{d}' . So Equation (5.6) hold also for densely many strengthenings of \bar{d}'' in $\mathbf{V}_{\alpha+1}$. So $\bar{d}'' \in D_{\alpha+1}$.

There are no new instances of (I5) in this step, since s_α is not part of the “sticking out” requirement. So (I5) for $\mathbb{P}_{\alpha+1}$ is just the conjunction of the

instances of (I5) that are true by induction hypothesis. \square **5.3**

The next lemma is for all other successor steps. Again we have to work on (I4) and this time also on (I5).

Lemma 5.4. *For $\alpha \in \omega_2 \setminus S_1^2$ there is $\mathbb{Q}_\alpha = \mathbb{M}$, the full Matet forcing, and there are σ -centred $\mathbb{Q}_\alpha = \mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$ such that (I1) to (I5) are carried on from \mathbb{P}_α to $\mathbb{P}_{\alpha+1}$.*

Proof. We construct it in $\mathbf{V}^{\mathbb{P}_\alpha}$ as $\mathbb{Q}_\alpha = \mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$ so that the \bar{c}_ε are chosen with respect to the desired forcing effect of \mathbb{Q}_α .

We consider (I4): By induction hypothesis we can step up from $\bar{a} \in \mathbf{V}_\beta \cap D_\beta$, to some $\bar{a}' \sqsubseteq^* \bar{a}$ with $\bar{a}' \in D_\alpha$ and also in \mathbf{V}_α , $\bar{a}' \in D_\beta$. So we only have to show the step from α to $\alpha+1$ in (I4). In addition we interweave the tasks for (I5): The new instance of (I5) is: $(\forall i \in \alpha \setminus S_1^2)(\forall \mathcal{R} \in \mathbf{V}_i)(\forall R \in \mathcal{R})(s_\alpha \not\sqsubseteq^* Rs_i)$.

The entire list of properties of $\langle \bar{c}_\varepsilon : \varepsilon < \omega_1 \rangle$ is:

We fix a sequence $\langle (\xi_\varepsilon, \bar{b}_\varepsilon, \bar{d}_\varepsilon, R_\varepsilon, B_\varepsilon, \bar{e}_\varepsilon) : \varepsilon < \omega_1 \rangle \in \mathbf{V}_\alpha$, such that

- (a) $\xi_\varepsilon \in \alpha \setminus S_1^2$,
- (b) $\bar{b}_\varepsilon, \bar{d}_\varepsilon \in (\mathbb{F})^\omega \cap \mathbf{V}_\alpha$,
- (c) $R_\varepsilon \in (\mathcal{R}^*)^{\mathbf{V}_\alpha}$,
- (d) $B_\varepsilon \subseteq \mathbb{F}$,
- (e) \bar{e}_ε is a $\mathbb{M}(\bar{b}_\varepsilon)$ -name for a member of $(\mathbb{F})^\omega$,
- (f) every such tuple $(\xi_\varepsilon, \bar{b}_\varepsilon, \bar{d}_\varepsilon, R_\varepsilon, B_\varepsilon, \bar{e}_\varepsilon)$ appears in the sequence \aleph_1 times.

We now choose \bar{c}_ε by induction on $\varepsilon < \omega_1$ such that

- (α) If $\zeta < \varepsilon$ then $\bar{c}_\zeta \sqsupseteq^* \bar{c}_\varepsilon$.
- (β) (The forcing tasks) $\bar{c}_\varepsilon \in \mathbf{V}_\alpha$, $\bar{c}_\varepsilon \in D_\alpha$, and the forcing tasks

$$(\forall^\infty k)(c_{\varepsilon+1,k} \not\sqsubseteq R_\varepsilon s_{\xi_\varepsilon}).$$

- (γ) (The Eisworth tasks) $\omega \setminus R_\varepsilon(\text{set}(\bar{c}_{\varepsilon+1})) \in R_\varepsilon(\text{cl}^{\mathbf{V}_\alpha}(\mathcal{E}))$.
- (δ) (The Hindman tasks) $\text{FU}(\bar{c}_{\varepsilon+1})$ is included in B_ε or disjoint from B_ε .
- (ε) (The sealing tasks) If \bar{e}_ε is an $\mathbb{M}(\bar{c}_\varepsilon)$ -name the sequence $\bar{c}_{\varepsilon+1}$ is chosen according to (m2') in the proof of Lemma 2.10: If $\bar{d}_\varepsilon \in \mathbf{V}_\alpha \cap D_\alpha$, then D_α is dense below \bar{d}_ε also in $\mathbf{V}_{\alpha+1}$, that is, if \bar{e}_ε stays a \mathbb{Q}_α -name, then $\bar{c}_{\varepsilon+1} \in \mathcal{C}_\alpha$ secures

$$\begin{aligned} \mathbf{V}_{\alpha+1} \models & \left(\bar{e}_\varepsilon \not\sqsubseteq^* \bar{d}_\varepsilon \right. \\ & \vee (\exists \bar{e} \sqsubseteq^* \bar{e}_\varepsilon) (\bar{e} \in \mathbf{V}_\alpha \wedge (\forall i \in \alpha \setminus S_1^2) \\ & \left. (\forall R \in \mathcal{R}^* \cap \mathbf{V}_i) (\exists^\infty n) (e_n \not\sqsubseteq Rs_i) \right). \end{aligned}$$

We show that the construction can be performed:

Beginning: $\alpha \notin S_1^2$. So by induction hypothesis of (I4) applied to $\langle \{i\} : i \in \omega \rangle$ and \mathbf{V}_0 there is $\bar{c}_0 \in \mathbf{V}_\alpha$, $\bar{c}_0 \in D_\alpha$. Note that the existence of such

a \bar{c}_0 is by far not trivial. At limit steps we take almost a condensation $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\zeta$, $\zeta < \varepsilon$. At successor steps we strengthen \bar{c}_ε four times so to fulfil the forcing task, the Hindman task, the Eisworth task and the sealing task for index ε .

We show that the construction works: The first part of (I4) follows from the sealing tasks. For the second part of (I4), how to cook up a name? Assume that $\bar{a} \in \mathbf{V}_\alpha$ and $\bar{a} \in D_\alpha$. We show:

$$(5.7) \quad \Vdash_{\mathbb{Q}_\alpha} \bar{a} \in D_{\alpha+1}.$$

Since D_α is dense below \bar{a} in the sense of \mathbf{V}_α and since this is preserved, we can just all the time, given $\bar{b} \in \mathbf{V}_{\alpha+1}$ with $\bar{b} \sqsubseteq \bar{a}$ take $\bar{e} \in \mathbf{V}_\alpha$ such that $\bar{e} \sqsubseteq^* \bar{b}$. The sealing tasks secure that D_α is also in $\mathbf{V}_{\alpha+1}$ dense. Moreover, since Rs_α is unbounded over \mathbf{V}_α for any $R \in \mathcal{R}^* \cap \mathbf{V}_\alpha$, we also have

$$(\forall R \in \mathbf{V}_\alpha)(\exists^\infty n)(e_n \not\subseteq Rs_\alpha).$$

So Equation (5.7) is true and (I4) is carried on.

The new instance of (I5) is: $\gamma = \alpha$ and $\beta \in \alpha \setminus S_1^2$. We have to show $s_\alpha \not\sqsubseteq^* R's_\beta$ for every $R' \in \mathcal{R}^* \cap \mathbf{V}_\beta$. This is secured by $\bar{c}_\varepsilon \in D_\alpha$ and

$$(5.8) \quad \begin{array}{l} \mathbb{Q}_\alpha \Vdash \text{“}s_\alpha \text{ is a union} \\ \text{of infinitely many blocks of } \bar{c}_\varepsilon \text{ and a finite rest.”} \end{array}$$

and the forcing tasks in (β) of the list.

Instead of the σ -centred $\mathbb{M}(\bar{c}_\varepsilon : \varepsilon < \omega_1)$ we can, in the \aleph_1 - \aleph_2 -scenario, also just take the full Matet order \mathbb{M} as \mathbb{Q}_α : The inductive procedure is a density argument in the \sqsubseteq^* -order for the possible elements of $\mathcal{C} = \langle \bar{c}_\varepsilon : \varepsilon < \omega_1 \rangle$. The second components of conditions in \mathbb{M} meet any \sqsubseteq^* -dense set. So \mathbb{M} together with its generic s_α fulfils the list that is numbered in Greek letters. \square 5.4

Now we consider two kinds of limit steps, those with countable cofinality, and those with cofinalities ω_1 or ω_2 .

Lemma 5.5. *Let $\alpha = \lim_n \alpha_n$ be the limit of a strictly increasing sequence of ordinals in ω_2 . If for each n , $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha_n, \gamma \leq \alpha_n \rangle$ fulfil (I1) to (I5), then also the countable support limit $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils (I1) to (I5).*

Proof. For (I1) we use a well-known preservation theorem: The countable support limit of forcings preserves each P -point that is preserved by all approximations [9, Theorem 4.1]. We also use that the countable support limit of proper forcings is proper [32, III, 3.2]. There are no new instances of (I3) in steps of countable cofinality. We check property (I4) for the new instance $\beta \in \alpha \setminus S_1^2$ and $\alpha \in S_0^2$ itself. Let $\alpha = \lim \alpha_n$, $\alpha_0 = \beta$, $\alpha_n \in \alpha \setminus S_1^2$. We let $\bar{a}_0 = \bar{a}$, $\bar{a} \in D_\beta$. We climb \sqsubseteq^* -downwards step for step with (I4) between α_n and α_{n+1} and get $\bar{a}_{n+1} \sqsubseteq^* \bar{a}_n$ and $\bar{a}_{n+1} \in D_{\alpha_{n+1}}$. In the end we let \bar{b} be almost a condensation of \bar{a}_n , $n \in \omega$.

By [2, Theorem 6.1.18], in $\mathbf{V}^{\mathbb{P}_\alpha}$ we have: D_β is dense below \bar{a} and each D_{α_n} is dense below \bar{a}_n . We use the upwards closure: If $\bar{a} \in D_\gamma$ and $\bar{a}' \sqsubseteq^* \bar{a}$, then $(\forall i \in \gamma \setminus S_1^2)(\forall R \in \mathbf{V}_i)(\exists^\infty n)(a'_n \not\subseteq Rs_i)$. So we have for the condensation \bar{b} : $(\forall i \in \alpha \setminus S_1^2)(\forall R \in \mathbf{V}_i)(\exists^\infty n)(b_n \not\subseteq Rs_i)$. This holds also for densely many strengthenings of \bar{b} in \mathbf{V}_α since we can repeat the climbing down ω many steps with starting point \bar{b} instead of \bar{a} . So in \mathbf{V}_α , $\bar{b} \in D_\alpha$.

There are no new instances of (I5) in limit steps of uncountable cofinality. \square

Lemma 5.6. *Let $\alpha = \lim_{\varepsilon < \mu} \alpha_\varepsilon$ be the limit of a strictly increasing sequence of ordinals in ω_2 and let μ be ω_1 or ω_2 . If for all ε , $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha_\varepsilon, \gamma \leq \alpha_\varepsilon \rangle$ fulfils (I1) to (I5), then also $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils (I1) to (I5).*

Proof. For (I1) and (I2) we invoke the same citations as in the previous lemma. Property (I3), (I4) and (I5) are carried on to limit steps of uncountable cofinalities since they are just a conjunction of conditions on (pairs of) strictly lower ordinals. There is no requirement on D_α for $\text{cf}(\alpha) \geq \omega_1$. \square

So we have proved the main theorem.

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REFERENCES

- [1] Uri Abraham. Proper forcing. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Kluwer, 2010.
- [2] Tomek Bartoszyński and Haim Judah. *Set Theory, On the Structure of the Real Line*. A K Peters, 1995.
- [3] Andreas Blass. Ultrafilters related to Hindman's finite unions theorem and its extensions. In Stephen Simpson, editor, *Logic and Combinatorics, Arcata, California, 1985*, volume 65 of *Contemp. Math.*, pages 89–124. Amer. Math. Soc., 1987.
- [4] Andreas Blass. Applications of superperfect forcing and its relatives. In Juris Steprāns and Steve Watson, editors, *Set Theory and its Applications*, volume 1401 of *Lecture Notes in Mathematics*, pages 18–40, 1989.

- [5] Andreas Blass. Groupwise density and related cardinals. *Arch. Math. Logic*, 30:1–11, 1990.
- [6] Andreas Blass. Homogeneous sets from several ultrafilters. *Topology Appl.*, 156:2581–2594, 2009.
- [7] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Kluwer, 2010.
- [8] Andreas Blass and Claude Laflamme. Consistency results about filters and the number of inequivalent growth types. *J. Symbolic Logic*, 54:50–56, 1989.
- [9] Andreas Blass and Saharon Shelah. There may be simple P_{\aleph_1} - and P_{\aleph_2} -points and the Rudin-Keisler ordering may be downward directed. *Ann. Pure Appl. Logic*, 33:213–243, 1987.
- [10] Andreas Blass and Saharon Shelah. Near coherence of filters III: A simplified consistency proof. *Notre Dame J. Formal Logic*, 30:530–538, 1989.
- [11] Jörg Brendle. Mob families and mad families. *Arch. Math. Logic*, 37(3):183–197, 1997.
- [12] Jörg Brendle. Distinguishing groupwise density numbers. *Monatshefte für Mathematik*, 152(3):207–215, 2007.
- [13] R. Michael Canjar. Mathias forcing which does not add dominating reals. *Proc. Amer. Math. Soc.*, 104:1239–1248, 1988.
- [14] Todd Eisworth. Forcing and stable ordered-union ultrafilters. *J. Symbolic Logic*, 67:449–464, 2002.
- [15] Vera Fischer and Juris Steprāns. The consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$. *Fund. Math.*, 2001(3):283–293, 2008.
- [16] Neil Hindman. Finite sums from sequences within cells of a partition of N . *J. Combin. Theory Ser. A*, 17:1–11, 1974.
- [17] Winfried Just and Claude Laflamme. Classifying sets of measure zero with respect to their open covers. *Trans. Amer. Math. Soc.*, 321(2):621–645, 1990.
- [18] Jakob Kellner and Saharon Shelah. Preserving preservation. *J. Symbolic Logic*, 70(3):914–945, 2005.
- [19] Jussi Ketonen. On the existence of P -points in the Stone-Čech compactification of integers. *Fund. Math.*, 92:91–94, 1976.
- [20] Kenneth Kunen. *Set Theory, An Introduction to Independence Proofs*. North-Holland, 1980.
- [21] Claude Laflamme. Equivalence of families of functions on natural numbers. *Trans. Amer. Math. Soc.*, 330:307–319, 1992.
- [22] Paul B. Larson. The filter dichotomy and medial limits. *J. Math. Log.*, 9(2):159–165, 2009.
- [23] Pierre Matet. Partitions and filters. *J. Symbolic Logic*, 51:12–21, 1986.
- [24] Heike Mildenberger. Groupwise dense families. *Arch. Math. Logic*, 40:93–112, 2001.
- [25] Heike Mildenberger and Saharon Shelah. Increasing the groupwise density number by c.c.c. forcing, [MdSh:843]. *Ann. Pure Appl. Logic*, 149:7–13, 2007.
- [26] Heike Mildenberger and Saharon Shelah. The principle of near coherence of filters does not imply the filter dichotomy principle [MdSh:894]. *Trans. Amer. Math. Soc.*, 361:2305–2317, 2009.
- [27] Heike Mildenberger and Saharon Shelah. The minimal cofinality of an ultrafilter of ω and the cofinality of the symmetric group can be larger than \mathfrak{b}^+ . *J. Symbolic Logic*, 76:1322–1340, 2011.
- [28] Heike Mildenberger, Saharon Shelah, and Boaz Tsaban. Covering the Baire space with meager sets. *Ann. Pure Appl. Logic*, 140:60–71, 2006.
- [29] Arnold Miller. There are no Q -points in Laver’s model for the Borel conjecture. *Proc. Amer. Math. Soc.*, 78:103–106, 1980.
- [30] Dilip Raghavan. Maximal almost disjoint families of functions. *Fund. Math.*, 204(3):241–282, 2009.

- [31] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, xxix+496 pp, 1982.
- [32] Saharon Shelah. *Proper and Improper Forcing, 2nd Edition*. Springer, 1998.
- [33] Saharon Shelah. Groupwise density cannot be much bigger than the unbounded number. *Math. Logic Quart.*, 54:340–344, 2008.
- [34] Michel Talagrand. Compacts de fonctions mesurables et filtres non mesurables. *Studia Mathematicae*, 67:13 – 43, 1980.

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